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On Complex Zeros of the q-Potts Partition Function for a Self-dual Family of Graphs

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Abstract This paper deals with the location of the complex zeros of q-Potts partition function for a class of self-dual graphs. For this class of graphs, as the form of the eigenvalues is known, the regions of the complex plane can be focused on the sets where there is only one dominant eigenvalue in particular containing the positive half plane. Thus, in these regions, the analyticity of the free energy per site can be derived easily. Next, some examples of graphs with their Tutte polynomial having few eigenvalues are given. The case of the cycle with an edge having a high order of multiplicity is presented in detail. In particular, we show that the well known conjecture of Chen et al. is false in the finite case. Furthermore we obtain a sequence of self-dual graphs for which the unit circle does not belong to the accumulation sets of the zeros.

Keywords Tutte polynomial · q-Potts model · Self-dual graph · Analytic function

1 Introduction

The q-state Potts model has served as a valuable model for the study of phase transitions and critical phenomena. On a connected graph $G=(V,E)$, where V is its vertex set and E its edge set, at temperature T, this model is defined by the partition function

$$Z(G,q,v)=\sum_{\{\sigma_n\}} e^{-\beta H}$$

with the zero field Hamiltonian

$$H=-J\sum_{\langle i,j\rangle}\delta_{\sigma_i\sigma_j}$$

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with J is the spin-spin coupling, $\sigma_i = 1 \dots q$ are the spin variables on each vertex $i \in V$; $\beta = (k_B T)^{-1}$; where k_B is the Boltzmann constant and $\langle i, j \rangle$ denotes pairs of adjacent vertices.

We use the following notation $v = e^{J/(k_B T)} - 1$, so that the physical ranges are $v \geq 0 \Leftrightarrow 0 \leq T, J \geq 0$, for the Potts ferromagnet and $-1 \leq v \leq 0 \Leftrightarrow 0 \leq T, J \leq 0$, for the Potts antiferromagnet.

For a graph $G = (V, E)$, we denote the number of vertices of G as $n = |V|$ and the number of edges of G as $e(G) = |E|$. Let define the free energy per site f:

$$f(G, q, v) = \lim_{n \to \infty} \frac{\ln(Z(G, q, v))}{n},$$

where $\{G\}$ denotes $\lim_{n \to \infty} G$ for a given family of graphs.

The partition function can be written as the sum

$$Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{e(G')},$$

where $G' = (V, E')$ is a subgraph of G having the same vertex set V and a subset of the edge set $E' \subseteq E$ and $k(G')$ denotes the number of connected components of G'. The Potts model partition function on a graph G is essentially equivalent to the Tutte polynomial $T(G, x, y)$:

$$Z(G, q, v) = (x - 1)(y - 1)^n T(G, x, y) \tag{1}$$

on the curve

$$(x - 1)(y - 1) = q \quad \Longleftrightarrow \quad \begin{cases} x = 1 + q/v, \\ y = 1 + v. \end{cases}$$

We denote $w = v\sqrt{q}$. For a planar G, the partition function possesses the duality relation:

$$Z(G, q, w) = q^{n-1-|E|/2} w^{e(G)} Z(G^\star, q, w^{-1}) \tag{2}$$

where $G^\star$ is the dual graph of G. Notice that such kind of duality relations are sources of inspiration for exact results in statistical mechanics [6, 10, 11], and for some correlation duality relations, see [9]. In the case of the square lattice for which $G^\star = G$ (i.e. G is self-dual) in the thermodynamic limit regardless of boundary conditions, (2) implies that the system is critical at $w_c = 1$. For $q = 1$ and exact independent bond percolation see [8] and the case $q = 2$ is the famous Ising model [13]. To take full advantage of the duality relation (2), we only consider a family of planar graphs which are self-duals for any value of n.

The program of studying complex zeros of the partition function was pioneered by Yang and Lee [18]. The location of the complex zeros of the partition function is important from a statistical mechanic point of view because it is related to a possible phase transition for q-Potts model on this family of graphs using the Fortuin Kasteleyn representation. When a complex region is free of zeros of the partition function, an exponential decay of correlation functions is deduced [14]: recall that correlation functions are also useful tools for the uniqueness of Gibbs models using Kirkwood-Salsburg equations [15].

We have several motivations. First, new exact calculations of Potts model partition functions on lattices strips with arbitrary large number of vertices are of value in their own right. This is because the free energy of Potts model has never been calculated exactly for $d \geq 2$ except in the Ising case in two dimensions [13]. The investigation of complex temperature phase diagrams of spin models can provide further understanding of the physical behavior of these models. In particular, we can find several works on exact determinations of complex-temperature phase diagrams of the Potts model on infinite length, finite width lattice strips. It shows that, although the physical thermodynamic properties of these strips are essentially one-dimensional, one can nevertheless gain important insights into certain complex-temperature properties of the model on the corresponding two dimensional lattice.

There exists an impressive number of papers concerning the location of zeros of q-Potts partition function: for instance, see [11] and [16] and references therein for general results for Potts model, and [3–5, 12] for different self-dual families of graphs. Recently Sokal [17] proves for general graphs that these zeros are dense in the complex plane. A large number of conjectures for a wide class of graphs such as triangulation, planar, cubic graphs are given in [7].

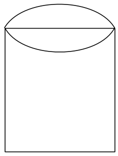
For self-dual graphs, an intriguing conjecture proposed by Chen et al. [5] asserts that, in a half plane, the complex zeros are located on a circle.

First of all, our goal was to prove the conjecture of Chen et al. [5] for the largest class of self-dual graphs as possible. In the finite case, this conjecture is true (see [3]) for the 1-strip graph with two different self-dual corrections. In fact as pointed in [5], the key seems to be the duality relation (2). More precisely, it was believed that it forced in the positive half plane the complex zeros of this polynomial to be located on the unit circle in the same spirit as the Lee and Yang theorem. We realized that it is not true. We give an example and surprisingly we find a sequence of self-dual graphs for which this unit circle does not belong to the accumulation sets of the zeros. But the question raised by this conjecture remains at the thermodynamic limit. Unfortunately in the general case, for example for self-dual strips of the square lattice, the eigenvalues coming from the powerful transfer matrix method are roots of polynomials with high degree: then, it is difficult to study the location of curves of degeneration of the dominant eigenvalue even if we focus on the positive half plane.

In this paper, we shall present exact calculations of the Potts model partition function $Z(G, q, w)$ on a family of self-dual graphs for which the form of the eigenvalues is known. To obtain this partition function, we compute the Tutte polynomial. We use deletion and contraction rules rather than the transfer-matrix method. We are able to present regions having only one dominant eigenvalue in particular containing the positive half plane. Then we derive easily in these regions of the complex plane the analyticity of the free energy per site. Next, we study some examples containing few eigenvalues for which degeneration cases can be completely described. Let us remark that even in the case of two or three eigenvalues as described in [3], we observed a rich variety of behavior for the accumulation sets of zeros in the w-complex plane. We choose to present here in details an example with two pairs of eigenvalues. One originality of this approach is to unify in the same framework different kind of duality preserving boundary conditions. Besides, we use a suitable variable allowing us to identify the set of accumulation of zeros more easily. But as a direct consequence of our result, the accumulation set of zeros in positive half plane is only a sector of the unit circle. It gives an analyticity result on the positive real axis as it is foreseeable for some essentially one dimensional graphs.

The outline of this paper is as following. First we give the conjecture of Chen et al. [5] and provide a counterexample. Next, the well-known Beraha Kahane Weiss theorem (see [1, 2]) is recalled. Then, general results showing regions where we have only one dominant eigenvalue, are established. The analyticity of the free energy per site is deduced. In a fourth section, we show particular sequences of self dual graphs that belong to our framework. For the cycle with an edge having a high order of multiplicity, the set of degeneration of the dominant eigenvalue is obtained. We conclude on possible extension of our work to other self-dual graphs.

Fig. 1 Graph C: a cycle with an edge of multiplicity three



2 Conjecture and Counterexample

In our case, we choose working in (1) with the variable

$$z = x + y - 2 = \sqrt{q}(w + 1/w) \quad \text{then } xy = z + q + 1.$$

It is easy to see that, on this curve and from the symmetry, the Tutte polynomial is also a polynomial of the variable z. Chen et al. [5] proposed the following conjecture:

For finite planar self-dual lattices and for square lattice with free or periodic boundary conditions in the thermodynamic limit, the Potts partition function zeros in the $\mathrm{Re}(w) > 0$ *half plane are located on the unit circle* $|w| = 1$.

This conjecture is not true in the finite case. Let us take the following simple example: choose the self-dual graph C with four vertices and six edges defined as a cycle of length four and an edge of multiplicity three, see Fig. 1. The Tutte polynomial of this graph on the previous hyperbola with $q = 16$ can be written as a function of z as

$$T(C, z) = z^3 + 6z^2 + 12z + 218.$$

There are one not positive real root $(-2 - (210)^{1/3})$ and two conjugated roots with not negative real part $((210)^{1/3}/2 - 2 \pm i(210)^{1/3}\sqrt{3}/2)$; then, from the relation between z and v, the complex roots in z with $\mathrm{Re}(z) > 0$ correspond with the roots in w in the region $\mathrm{Re}(w) > 0$ not on the unit circle. At least in the finite case, this conjecture is false.

3 General Results

3.1 Preliminary

At the thermodynamic limit, the analyticity of the free energy per site can be obtain by studying and avoiding the location of the accumulation sets of zeros of the partition function (or Tutte polynomial) for a family of self-dual graphs.

A central role in our work is played by a theorem on analytic functions due to Beraha, Kahane and Weiss.

Theorem 3.1 [1, 2] *Let D be a domain* (*connected open set*) *in $\mathbb{C}$ and let $\lambda_1, \ldots, \lambda_M$, $\alpha_1, \ldots, \alpha_M$ be analytic functions on D, none of which is identically zero. Let us further assume a no-degenerate-dominance condition*: *there do not exist subscripts $k \neq k'$ such that $\lambda_k = \omega\lambda_{k'}$ for some constant ω with $|\omega| = 1$ and such that $\{z \in D : k$ is dominant at $z\}$ $(= \{z \in D : k'$ is dominant at $z\})$ has nonempty interior. For each integer $n \geq 0$, define f_n by*

$$f_n(z) = \sum_{k=1}^{M} \alpha_k(z)\lambda_k(z)^n$$

and the zero sets $Z(f_n) = \{z \in D : f_n(z) = 0\}$.

Then $\{z \in D :$ *every neighborhood* $U \ni z$ *has a nonempty intersection with all but finitely many of the sets* $Z(f_n)\} = \{z \in D :$ *every neighborhood* $U \ni z$ *has a nonempty intersection with infinitely many of the sets* $Z(f_n)\}$, *and a point* z *lies in this set if and only if either*

(a) *There is a unique dominant subscript* k *at* z (*i.e.* $|\lambda_k(z)| > |\lambda_l(z)|$ *for all* $l \in \{1 \dots M\} \setminus \{k\}$) , *and* $\alpha_k(z) = 0$; *or*
(b) *There are two or more dominant subscripts at* z.

3.2 Analyticity of the Free Energy per Site

Here, we work with a family of self-dual graphs G_n for which the Tutte polynomials are of the following form:

$$f_n(z) = T(G_n, z) = \sum_{k=1}^{M} \Big[\alpha_{a_k}(z)[\lambda^+_{a_k}(z)]^n + \beta_{a_k}(z)[\lambda^-_{a_k}(z)]^n\Big]$$

where

$$\lambda^{\pm}_{a_k}(z) = \frac{1}{2}\left(z + 2 + a_k \pm \sqrt{(z + a_k + 2)^2 - 4(z + q + 1)}\right)$$

with $a_k \in [0, q]$.

By convention, we denote $\lambda_{a_k}(z)$, the function between $\lambda^+_{a_k}(z)$ and $\lambda^-_{a_k}(z)$ with the greatest magnitude. The functions $\{\alpha_{a_k}, \beta_{a_k}, k = 1..M\}$ are chosen such that $T(G_n, .)$ stays a polynomial function in the variable z. We denote by $a_u = \sup_{k=1\dots M} a_k$ and $a_l = \inf_{k=1\dots M} a_k$. For any real interval I, we also use some subset of the complex plane like:

$$D_I = \{z = c + id; (c, d) \in \mathbb{R}^2, c \in I\}.$$

Now taking $p_n(z) = \frac{\ln(T(G_n,z))}{n}$, the analyticity of the free energy per site in some regions can be derived easily:

Theorem 3.2 *There exists only one dominant eigenvalue at* z:

- $\forall z \in D_{]-a_l,+\infty[} \setminus \{c \in [-a_l, \sup(-a_l, -a_u + 2\sqrt{q - a_u})], d = 0\}$, $\lambda_{a_u}(z)$ *is the dominant eigenvalue and if* $\alpha_{a_u}(z) \neq 0$, *then* $p_n(z) \to \ln[\lambda_{a_u}(z)]$ *as* $n \to \infty$.
- $\forall z \in D_{]-\infty,-a_u-2[} \setminus \{c \in [\inf(-a_u - 2, -a_l - 2\sqrt{q - a_l}), -a_u - 2], d = 0\}$, $\lambda_{a_l}(z)$ *is the dominant eigenvalue and if* $\beta_{a_l}(z) \neq 0$, *then* $p_n(z) \to \ln[\lambda_{a_l}(z)]$ *as* $n \to \infty$.

Let K *be a non negative real and* $B((0, 0), K)$ *the disk center at* $(0, 0)$ *with radius* K. *Both limits are analytic functions of* z *respectively on the intersection of previous subsets with* $B((0, 0), K)$.

Proof By introducing

$$A = \mathrm{Re}((z + a + 2)^2 - 4(z + q + 1)) = (a + c)^2 - d^2 - 4(q - a),$$
$$B = |(z + a + 2)^2 - 4(z + q + 1)| = \sqrt{A^2 + 4d^2(a + c)^2},$$

the eigenvalues can be written under this form:

$$\begin{cases} \lambda^+_a(z) = \frac{1}{2}(a + c + 2 + (-1)^m V) + \frac{i}{2}(d + (-1)^p U), \\ \lambda^-_a(z) = \frac{1}{2}(a + c + 2 - (-1)^m V) + \frac{i}{2}(d - (-1)^p U), \end{cases}$$

where

$$\begin{cases} U = \sqrt{\frac{B-A}{2}}, \\ V = \sqrt{\frac{A+B}{2}}, \end{cases} \qquad m = \begin{cases} 0 & \text{if } a+c>0, \\ 1 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad p = \begin{cases} 0 & \text{if } d>0, \\ 1 & \text{otherwise.} \end{cases}$$

By differentiation the magnitude of $\lambda_a^+(z)$ in function of a, it comes

$$(2|\lambda_a^+(z)|^2)' = \frac{(-1)^m V}{B}\left\{\left[V\left(1+\frac{U(-1)^p}{d}\right)+2\right]^2 + \left[U+(-1)^p d\right]^2\right\}. \tag{3}$$

Hence it is enough to remark that not increasing or not decreasing depends only on the parity of m.

Now, we identify the eigenvalue between $\lambda_a^+(z)$ and $\lambda_a^-(z)$ with greatest modulus which will be denoted by $\lambda_a(z)$. It comes by the monotonicity of the eigenvalues,

$$\begin{cases} \text{if } d \neq 0,\, \lambda_a(z) = \lambda_a^+(z) \text{ on } D_{]-\infty,-a-2[} \cup D_{]-a,+\infty[}, \\ \text{if } d=0,\ \forall a \in [0, q-1],\, |\lambda_a^+(z)| = |\lambda_a^-(z)| \\ \quad \Longleftrightarrow \quad d=0,\ c \in [-a-2\sqrt{q-a}, -a+2\sqrt{q-a}], \\ \text{if } d=0,\ \forall a \in [q-1, q],\ |\lambda_a^+(z)| = |\lambda_a^-(z)| \\ \quad \Longleftrightarrow \quad \begin{cases} d=0,\ c \in [-a-2\sqrt{q-a}, -a+2\sqrt{q-a}] \\ \text{or } z \in C((-q-1,0), 1-q+a), \end{cases} \end{cases}$$

where $C((-q-1,0),1-q+a)$ is the circle of center $(-q-1,0)$ with radius $1-q+a$.

Then, two first points of the theorem can be deduced from previous system and (3).

Moreover, because of the given form of $\lambda_{a_u}(z)$,

$$1 \le |\lambda_{a_u}(z)| \le C_1(K)$$

with $C_1(K)$ is a constant depending only of K, a_u and q. We have also that

$$\forall z \in D_{]-\infty,-a_u-2[} \setminus \{c \in [\inf(-a_u-2, -a_l-2\sqrt{q-a_l}), -a_u-2], d=0\},$$

the dominant eigenvalue is λ_{a_l}. We find in this region that

$$\frac{a_u-a_l}{2} \le |\lambda_{a_l}(z)| \le C_2(K)$$

with $C_2(K)$ is a constant depending only of K, a_l and q. Now the result is a direct application of Vitali's convergence theorem. □

Remarks

(1) The case $d=0$ implies that, if $A>0$, the eigenvalues are reals and different. We can not have $|\lambda_a^+(z)| = |\lambda_a^-(z)|$. If $A<0$, the eigenvalues are conjugated and we have $|\lambda_a(z)| = |\lambda_a^-(z)| = |\lambda_a^+(z)|$.

(2) Given a Tutte polynomial written with M pairs of eigenvalues like $(\lambda_{a_k}^+(z), \lambda_{a_k}^-(z))$, $k=1\dots M$. The cases of degeneracy look like to $|\lambda_{a_k}^+(z)| = |\lambda_{a_k}^-(z)|$ or $|\lambda_{a_k}^+(z)| = |\lambda_{a_{k'}}^-(z)|$, for some k, k' in $\{1,\dots,M\}$, where these eigenvalues are dominant. As the

products $|\lambda_{a_k}^{+}(z)\lambda_{a_k}^{-}(z)|$ are constant, the first case implies that all eigenvalues have the same magnitude. The degeneracy region in z is given by

$$\bigcap_{k=1\ldots M}[-a_k-2\sqrt{q-a_k};-a_k+2\sqrt{q-a_k}].$$

For the second case, $|\lambda_a^{+}(z)|$ is a not increasing function of a, $a\in[a_l;a_u[$, on the set $D_{]-\infty;-a_u[}$ and a not decreasing function of a, $a\in[a_l;a_u[$, on the set $D_{]-a_l;+\infty[}$. Thus, we have to study only one case of degeneration of these sets:

$$|\lambda_{a_l}(z)|=|\lambda_{a_u}(z)|.$$

(3) Usually we work with the variable $w=re^{i\theta}$ such as $z=c+id=\sqrt{q}(w+1/w)$. We call F the transformation from z to w:

$$w=F(z)=\frac{z\pm\sqrt{z^2-4q}}{2\sqrt{q}}.$$

For example when we have only one pair of eigenvalues, we just need to calculate $F(z)$ when $z\in\{(c,d)\in\mathbb{R}^2, d=0,\ c\in[-a-2\sqrt{q-a},-a+2\sqrt{q-a}]\}$ and when z belongs to the circle $C((-q-1,0),1-q+a)$ for $q-1\le a\le q$.

4 Examples

4.1 Two Families of Self-dual 1-strip Graphs

After working in two specified regions of the complex z plane, it may be interesting to know what happens elsewhere. Several simple examples of self-dual graphs are given: the classical one is the wheel. The other graphs—the 1-strip graph with duality preserving boundary condition of type 1 (DBC1), the cycle with one multiple edge—are built from the wheel by moving some of its edges. By making this transformation, we only have to keep the symmetry property of the Tutte polynomial. These graphs belong to the framework we discussed before for a particular choice of the parameters a_k like 0, 1, q.

Let recall the results obtained by [3] on the wheel (also seen as 1-strip graph with DBC2) and for the 1-strip graph with DBC1. For the last one G_n with n triangles (see Fig. 2), on the hyperbola $(x-1)(y-1)=q$, the eigenvalues are of the form introduced before with $a=1$ and the Tutte polynomial can be written as

$$T(G_n,z)=\alpha_1(z)[\lambda_1^{+}(z)]^{n+1}+\beta_1(z)[\lambda_1^{-}(z)]^{n+1}$$

with $\alpha_1(z)=\frac{\lambda_1^{+}(z)-1}{\lambda_1^{+}(z)-\lambda_1^{-}(z)}$ and $\beta_1(z)=\frac{\lambda_1^{-}(z)-1}{\lambda_1^{-}(z)-\lambda_1^{+}(z)}$. For the wheel B_n with n triangles (see Fig. 2), on the hyperbola $(x-1)(y-1)=q$, these eigenvalues are of the form introduced

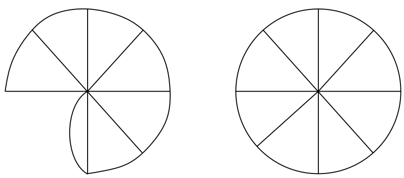


Fig. 2 G_n the 1-strip graph with DBC1, B_n the wheel (or 1-strip graph with DBC2)

before with $a = 1$ and $a = q$. The Tutte polynomial can be written as:

$$T(B_n, z) = (q-2)[\lambda_q^-(z)]^n + [\lambda_1^+(z)]^n + [\lambda_1^-(z)]^n$$

where $\lambda_q^-(z) = 1$. For both families of graphs, the determination of their Tutte polynomials and the degeneration cases have been already studied in [3].

4.2 Cycle with One Multiple Edge

In this section, we provide the Tutte polynomial associated with the cycle with one multiple edge using the contraction and deletion rules useful to obtain recurrence formula. We need these recurrence formula to identify without ambiguity the functions $\lambda_{a_k}^{\pm}$ introduced before. We present the location of accumulation sets of zeros and more precisely the curves describing the degeneration of the dominant eigenvalue using respectively the variable z and w in the whole complex plane.

We consider graphs having $2n$ edges. We assume that, for such a graph, we have one cycle C_{n+1} of length $n+1$ and one edge with order of multiplicity $k = n$. We denote these graphs by $G_{n,k}$.

Let us introduce the following graphs $A_{n,k}$, $B_{n,k}$ and $C_{n,k}$: $A_{n,k}$ is the graph having one cycle of length n and one vertex having k loops; $B_{n,k}$ is the graph having one isthmus on length n and ended by k loops; and $C_{n,k}$ is the graph having one isthmus of length $n+1$ and the last edge with order of multiplicity k. These graphs are presented in Fig. 3.

For all graphs G, the notation $T(G)$ for the Tutte polynomial $T(G, x, y)$ evaluated at the point (x, y) shall be used. We have

$$\begin{aligned}
T(G_{n,k}) &= T(C_{n-1,k}) + T(G_{n-1,k}) \\
&= T(C_{n-1,k-1}) + T(B_{n-1,k-1}) + T(G_{n-1,k}) \\
&= xT(C_{n-2,k-1}) + T(B_{n-1,k-1}) + T(G_{n-1,k}) \\
&= x[T(G_{n-1,k-1}) - T(G_{n-2,k-1})] + T(B_{n-1,k-1}) + T(G_{n-1,k}) \\
&= x[T(G_{n-1,k-1}) - T(G_{n-2,k-2} - T(A_{n-2,k-2})] \\
&\quad + T(B_{n-1,k-1}) + T(G_{n-1,k})
\end{aligned}$$

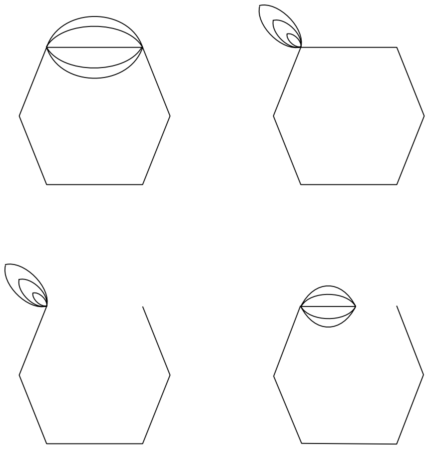


Fig. 3 From left to right and from top to bottom, the graphs $G_{n,k}$, $A_{n,k}$, $B_{n,k}$ and $C_{n,k}$

then

$$T(G_{n,k}) = x[T(G_{n-1,k-1}) - T(G_{n-2,k-2})] - xT(A_{n-2,k-2}) \\ + xyT(B_{n-2,k-2}) + T(G_{n-1,k-1}) + T(A_{n-1,k-1}).$$

We deduce that

$$T(G_{n,k}) - T(G_{n-1,k-1}) = x[T(G_{n-1,k-1}) - T(G_{n-2,k-2})] \\ + (y-x)T(A_{n-2,k-2}) + y(x+1)T(B_{n-2,k-2}).$$

Moreover, we have both following relations

$$T(A_{n-1,k-1}) = yT(A_{n-2,k-2}) + yT(B_{n-2,k-2}), \tag{4}$$

$$T(B_{n-1,k-1}) = xyT(B_{n-2,k-2}). \tag{5}$$

Taking $U_{n,k} = T(G_{n,k}) - T(G_{n-1,k-1})$, it comes that

$$U_{n,k} = xU_{n-1,k-1} + (y-x)T(A_{n-2,k-2}) + y(x+1)T(B_{n-2,k-2}), \tag{6}$$

$$U_{n-1,k-1} = xU_{n-2,k-2} + (y-x)T(A_{n-3,k-3}) + y(x+1)T(B_{n-3,k-3}). \tag{7}$$

Using the relation (4) and computing (6)–(7), we obtain:

$$U_{n,k} - (x+y)U_{n-1,k-1} + xyU_{n-2,k-2} = (xy-1)T(B_{n-2,k-2}).$$

Considering this last relation with order $n-1, k-1$, (5) and after subtracting and multiplying by xy, it leads to

$$U_{n,k} - (x+y+xy)U_{n-1,k-1} + xy(1+x+y)U_{n-2,k-2} - (xy)^2 U_{n-3,k-3} = 0.$$

Now, writing with the help of previous Tutte polynomials, we find the following relation:

$$T(G_{n,n}) = \gamma T(G_{n-1,n-1}) - (\gamma - 1 + xy(1+x+y))T(G_{n-2,n-2}) \\ + \gamma xyT(G_{n-3,n-3}) - (xy)^2 T(G_{n-4,n-4})$$

where $\gamma = xy + x + y + 1$. Hence,

$$\begin{pmatrix} T(G_{n,n}) \\ T(G_{n-1,n-1}) \\ T(G_{n-2,n-2}) \\ T(G_{n-3,n-3}) \end{pmatrix} = \begin{pmatrix} \gamma & -(\gamma - 1 + xy(1+x+y)) & \gamma xy & -(xy)^2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} T(G_{n-1,n-1}) \\ T(G_{n-2,n-2}) \\ T(G_{n-3,n-3}) \\ T(G_{n-4,n-4}) \end{pmatrix},$$

$$\begin{pmatrix} T(G_{n,n}) \\ T(G_{n-1,n-1}) \\ T(G_{n-2,n-2}) \\ T(G_{n-3,n-3}) \end{pmatrix} = P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & xy \end{pmatrix}^{n-3} P^{-1} \begin{pmatrix} T(G_{3,3}) \\ T(G_{2,2}) \\ T(G_{1,1}) \\ T(G_{0,0}) \end{pmatrix},$$

where

$$T(G_{3,3}) = x^3 + y^3 + xy^2 + x^2y + x^2y^2 + x^2 + y^2 + xy + x + y,$$

$$T(G_{2,2}) = x^2 + y^2 + xy + x + y, \qquad T(G_{1,1}) = x + y, \qquad T(G_{0,0}) = 1.$$

Then, the Tutte polynomial is given by

$$T(G_{n,n}) = \frac{xy - x - y}{(x-1)(y-1)}[x^n + y^n - 1] + \frac{(xy)^n}{(x-1)(y-1)}.$$

On the hyperbola $(x-1)(y-1) = q$, these eigenvalues are of the form introduced before with $a = 0$ for $\{x, y\}$ and with $a = q$ for $\{1, xy\}$. The Tutte polynomial can be written as:

$$T(G_{n,n}, z) = \frac{1}{q}[\lambda_q^+(z)]^n + \frac{1-q}{q}[\lambda_q^-(z)]^n + \frac{q-1}{q}[\lambda_0^+(z)]^n + \frac{q-1}{q}[\lambda_0^-(z)]^n.$$

Proposition 4.1 *For the family of graphs $(G_{n,n})_{n\geq 0}$, the location of the degeneration of the dominant eigenvalue is described in the complex plane using respectively variable z and w as follows*:

$$\begin{aligned} &\forall q \geq 1,\ \textit{when}\ |\lambda_0(z)| = |\lambda_q(z)| \\ &\iff \left\{(c,d) \in \mathbb{R}^2, c \in [-2 - q/2, -q/2], d^2 = -(2c + q + 4)\left[\frac{(c+q)^2}{2c+q}\right]\right\} \\ &\iff w \in \Delta_{-\sqrt{q}/2} \cup C((-1/\sqrt{q}, 0), 1/\sqrt{q}) \end{aligned}$$

where $\Delta_{-\sqrt{q}/2}$ is the line $\mathrm{Re}(w) = -\sqrt{q}/2$ *and $C((-1/\sqrt{q}, 0), 1/\sqrt{q})$ denotes the circle of center $(-1/\sqrt{q}, 0)$ and of radius $1/\sqrt{q}$.*

Proof First, we have that

$$|\lambda_a(z)| = |\lambda_q(z)| \iff \begin{cases} c \in [-(q+4+a)/2, -(q+a)/2], \\ d^2 = -(c+q)^2 \frac{2c+q+4+a}{2c+q+a}. \end{cases}$$

Then, just take $a = 0$. Now, using variable w, it is easy to see that $|y| = |1 + \sqrt{q}w| = 1$ leads to $w \in C((-1/\sqrt{q}, 0), 1/\sqrt{q})$ and $|x| = |1 + \sqrt{q}/w| = 1$ to $w \in \Delta_{-\sqrt{q}/2}$. □

We notice that, for $q > 4$ or $-q < -2\sqrt{q}$, then the unit circle $|w| = 1$ is not at all in the set of degeneration of the dominant eigenvalue. This is completely the opposite of what we are waiting for in the case of self-dual strip of the finite square lattice. However, it comes that the w-complex positive half plane does not contain any accumulation set of zeros.

5 Concluding Remark

We can find many other self-dual graphs belonging to this framework: these graphs can be seen as a transition between the wheel and the cycle with a multiple edge. Moreover, we point out that other choice of parameter a might be interesting. For example consider a family of self-dual strip graphs defined as follows. Let the $L_x \times L_y$ lattice strip have periodic boundary longitudinal or horizontal boundary condition and connect all of the vertices on the upper side of the strip to a single external vertex, while all of the vertices on the lower side of the strip have a free boundary condition. The case $L_y = 1$ is the wheel graph discussed previously. For this family of graphs, the Tutte polynomial is already known; in particular, for the last but one transfer matrices, the eigenvalues obtained for $L_y = 2$ and $L_y = 3$ are respectively roots of the following equations

$$\begin{aligned}\lambda^4 &- (2(x+y)+3)\lambda^3 + (x^2+y^2+3(x+y)+4xy+1)\lambda^2 \\ &- xy(2(x+y)+3)\lambda + (xy)^2 = 0\end{aligned}$$

and

$$\begin{aligned}\lambda^6 &- (3(x+y)+5)\lambda^5 + (3(x^2+y^2)+10(x+y)+9xy+6)\lambda^4 \\ &- (x^3+y^3+5(x^2+y^2)+9xy(x+y)+20xy+6(x+y))\lambda^3 \\ &+ xy(3(x^2+y^2)+10(x+y)+9xy+6)\lambda^2 - x^2y^2(3(x+y)+5)\lambda + (xy)^3 = 0.\end{aligned}$$

For this kind of graphs, the complex-temperature phase diagrams have been already computed for example in [3]. For these equations, we find eigenvalues of the form we have studied in this paper. Denoting by $b_r = 2 + 2\cos(2\pi/r)$, the Tutte Beraha numbers, we just have to choose values of $a = b_5, 3 - b_5$ for $L_y = 2$ and $a = b_7, 2 - \sqrt{b_7}, 3 - b_7 + \sqrt{b_7}$ for $L_y = 3$. Unfortunately, it seems that the eigenvalues of others transfer matrices were not of the desired form. It would be interesting to find what choice of parameters a, α_a, β_a allows to include other known family of self-dual graphs.

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